Binary Relation II

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Adapted From:
Stanford CS103 Binary Relation
《数理逻辑与集合论》10.4-10.8

Review

- **Binary Relation Basics**
  - Subset of Cartesian Products
  - Representation: Relation matrix and Digraph
  - Operations: Union, composite, power, etc

Review: Power of Relations

$\boldsymbol{A}$ is a finite set and $|\boldsymbol{A}| = n$, $\boldsymbol{R}$ is a relation on $\boldsymbol{A}$, then there exits natural number s, t , $s \neq t$,such that $R^s = R^t$

Proof:

Any relation on $\boldsymbol{A}$ is a subset of $\boldsymbol{A} \times \boldsymbol{A}$, or an element of $\mathcal{P}(A \times A)$.

We know that $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{|A|^2} = 2^{n^2}$, so there are finite many relations on a finite set $\boldsymbol{A}$. But there are infinite many powers of R. It means there are different powers of $\boldsymbol{R}$ which are equal, or

$$(\exists s)(\exists t)(s \in \mathbb{N} \land t \in \mathbb{N} \land s \neq t \land R^s = R^t)$$

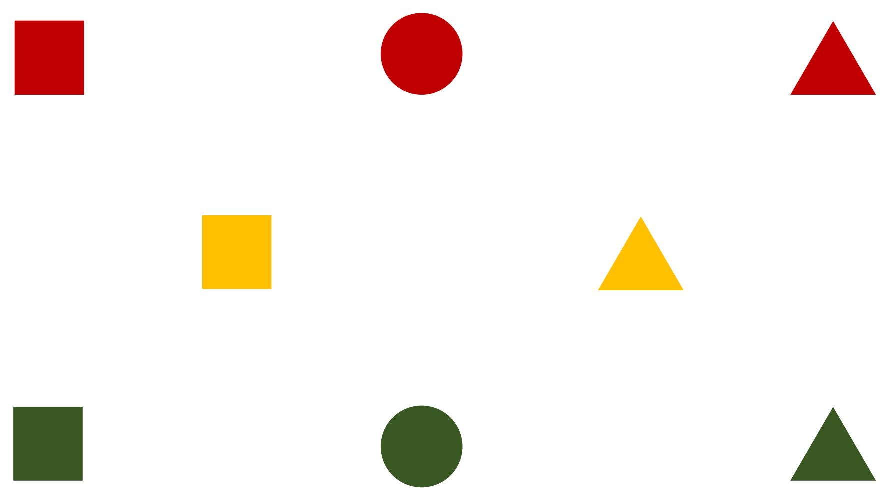
Q.E.D

Review: Power of Relations

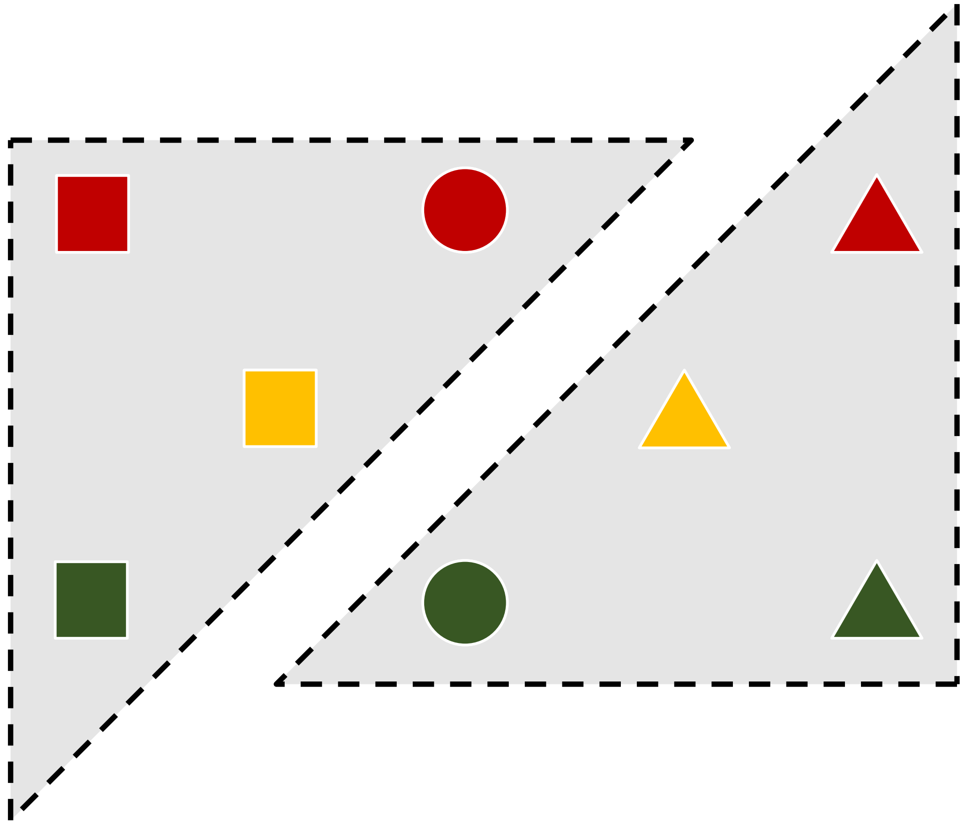
- $R^m \circ R^n = R^{m+n}$
- $(R^m)^n = R^{mn}$
- $\boldsymbol{A}$ is a finite set and $|\boldsymbol{A}| = n$, $\boldsymbol{R}$ is a relation on $\boldsymbol{A}$, and there exits natural number s, t , $s \neq t$,such that $R^s = R^t$
  - $R^{s+k} = R^{t+\mathrm{k}}, k \in \mathbb{N}$
  - $R^{s+kp+i} = R^{t+\mathrm{k}p+i}, k \in \mathbb{N}, i \in \mathbb{N}, p = t - s$
  - Let $B = \{R^0, R^1, \dots R^{t-1}\}$, then for any natural number q, $R^q \in B$

How can we capture a structure with relations?

Partitions(划分)



Partitions



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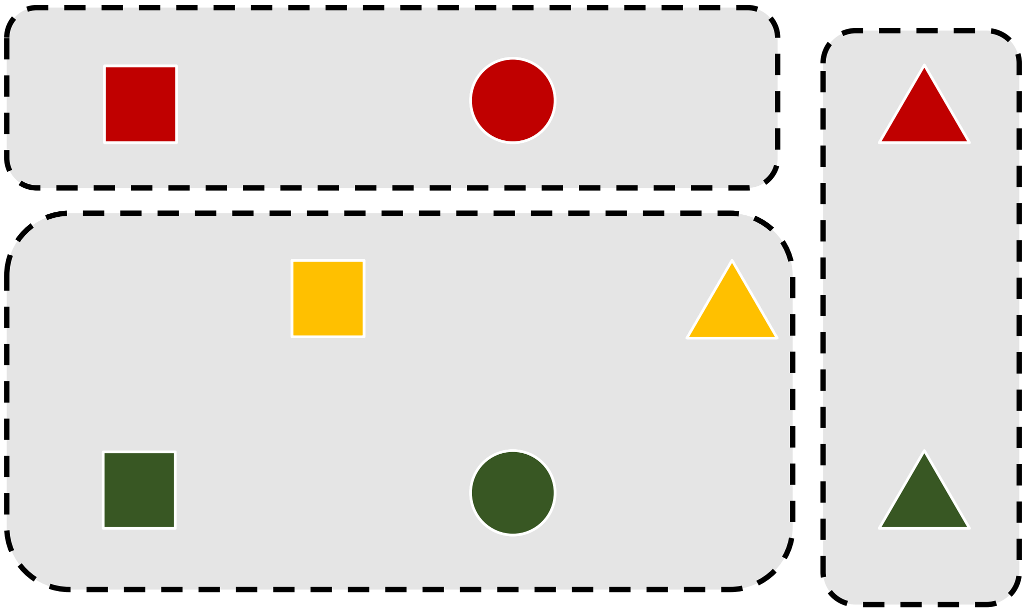
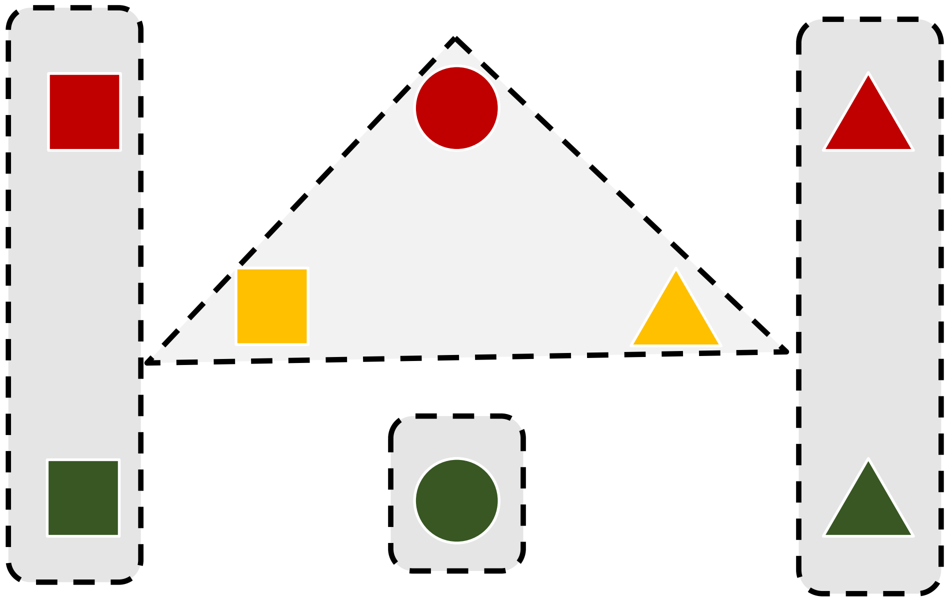


Partitions



Partitions

- A **partition of a set** is a way of splitting the set into **disjoint, nonempty subsets** so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.

Partitions

- More formally, for a non-empty set $\boldsymbol{A}$, a set $\boldsymbol{\pi}$ is called the partition of $\boldsymbol{A}$ if:
  - $\emptyset \notin \boldsymbol{\pi}$
  - $(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{\pi} \to \boldsymbol{x} \subseteq \boldsymbol{A})$
  - $\bigcup \boldsymbol{\pi} = \boldsymbol{A}$
  - $(\forall \boldsymbol{x})(\forall \boldsymbol{y})((\boldsymbol{x} \in \boldsymbol{\pi} \wedge \boldsymbol{y} \in \boldsymbol{\pi} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \to \boldsymbol{x} \cap \boldsymbol{y} = \emptyset)$

Partitions

- For set $A = \{a, b, c, d\}$
  - $\boldsymbol{\pi_1} = \{\{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}, \{\boldsymbol{d}\}\}$
  - $\boldsymbol{\pi_2} = \{\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}\}$

valid partitions

  - $\boldsymbol{\pi_3} = \{\{\boldsymbol{a}, \boldsymbol{b}\}, \{\boldsymbol{c}\}, \{\boldsymbol{a}, \boldsymbol{d}\}\}$
  - $\boldsymbol{\pi_4} = \{\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}\}$

not partition

What is the relationship between partitions and relations?

Relation in Partition

- For a partition $\boldsymbol{\pi}$ of set $\boldsymbol{A}$, we can define a relation R such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same partition block.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{\pi} \wedge \boldsymbol{a} \in \boldsymbol{\pi_0} \wedge \boldsymbol{b} \in \boldsymbol{\pi_0})\}$$

- Then we have:
  - $\boldsymbol{aRa}$ **for** $\forall \boldsymbol{a} \in \boldsymbol{A}$
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$
  - If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$

Relation in Partition

- For a partition $\boldsymbol{\pi}$ of set $\boldsymbol{A}$, we can define a relation R such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same partition block.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{\pi} \land \boldsymbol{a} \in \boldsymbol{\pi_0} \land \boldsymbol{b} \in \boldsymbol{\pi_0})\}$$

- Then we have:
  - $\boldsymbol{aRa}$ **for** $\forall \boldsymbol{a} \in \boldsymbol{A}$

    An element is in the same partition block with itself.
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$

    If **a** and **b** are in the same block, then **b** and **b** are in the same block too.
  - If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$

    If **a** and **b** are in the same block, **b** and **c** are also in the same block, then **a** and **c** are in the same block which contains **b**.

Relation in Partition

$$(\forall a)(a \in A \to aRa)$$

$$(\forall a)(\forall b)((a \in A \land b \in A \land aRb) \to bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$\big((a \in A \land b \in A \land c \in A \land aRb \land bRc) \to aRc\big)$$

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \land b \in A \land aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$((a \in A \land b \in A \land c \in A \land aRb \land bRc) \rightarrow aRc)$$

Reflexivity(自反性)

- Some relations always hold from any element to itself.
- Examples:
  - $\boldsymbol{x} = \boldsymbol{x}$ for any x
  - $\boldsymbol{A} \subseteq \boldsymbol{A}$ for any set A
  - $\boldsymbol{x} \geq \boldsymbol{x}$ for any x
- Relations of this sort are called reflexive.

$$\boldsymbol{R} \textbf{ on } \boldsymbol{A} \textbf{ is reflexive} \Leftrightarrow (\forall \boldsymbol{a})(\boldsymbol{a} \in \boldsymbol{A} \rightarrow \boldsymbol{aRa})$$

Reflexivity: Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $=, \geq, \leq$ on $\mathbb{R}$
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$ on set $\boldsymbol{A} = \{1,2,3\}$

Reflexivity: Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$ ☺
  - aI_Aa and aE_Aa for all $a \in A$
- $=, \geq, \leq$ on $\mathbb{R}$ ☺
  - $a = a, a \geq a, a \leq a$ for all $a \in \mathbb{R}$
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle\}$ on set $\boldsymbol{A} = \{1,2,3\}$ ☹
  - $\langle 3,3\rangle \notin R$, it's not reflexive

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \land b \in A \land aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$((a \in A \land b \in A \land c \in A \land aRb \land bRc) \rightarrow aRc)$$

Symmetry(对称性)

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $\boldsymbol{x} = \boldsymbol{y}$ then $\boldsymbol{y} = \boldsymbol{x}$
  - If $\boldsymbol{x} \equiv_{\boldsymbol{k}} \boldsymbol{y}$ then $\boldsymbol{y} \equiv_{\boldsymbol{k}} \boldsymbol{x}$
- These relations are called symmetric.

$$\boldsymbol{R} \textbf{ on } \boldsymbol{A} \textbf{ is symmetric} \Leftrightarrow$$
$$(\forall \boldsymbol{a})(\forall \boldsymbol{b})((\boldsymbol{a} \in \boldsymbol{A} \land \boldsymbol{b} \in \boldsymbol{A} \land \boldsymbol{aRb}) \to \boldsymbol{bRa})$$

Symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $\boldsymbol{N_A}$
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$ ☺
  - $aI_Ab \to bI_Aa$, $aE_Ab \to bE_Aa$
- $\boldsymbol{N_A}$ ☺
  - $aN_Ab \to bN_Aa$ always hold cause aN_Ab never holds
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$ ☺

Symmetry : Relation Matrix

$\boldsymbol{R}$ is symmetric $\Leftrightarrow$ $\boldsymbol{M_R}$ is symmetric matrix

- So if $\boldsymbol{R}$ is symmetric , then $\boldsymbol{R^{-1} = R}$ as $\boldsymbol{M_{R^{-1}} = M_R^T = M_R}$
- $\boldsymbol{M_{E_A}}$, $\boldsymbol{M_{I_A}}$ and $\boldsymbol{M_{N_A}}$ are both symmetric matrix

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c)$$
$$\big((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc\big)$$

Transitivity(传递性)

- Many relations can be chained together.
- Examples:
  - If $\boldsymbol{x} = \boldsymbol{y}$ and $\boldsymbol{y} = \boldsymbol{z}$ then $\boldsymbol{x} = \boldsymbol{z}$
  - If $\boldsymbol{P} \subseteq \boldsymbol{Q}$ and $\boldsymbol{Q} \subseteq \boldsymbol{S}$, then $\boldsymbol{P} \subseteq \boldsymbol{S}$
  - If $\boldsymbol{x} \equiv_{\boldsymbol{k}} \boldsymbol{y}$ and $\boldsymbol{y} \equiv_{\boldsymbol{k}} \boldsymbol{z}$, then $\boldsymbol{x} \equiv_{\boldsymbol{k}} \boldsymbol{z}$
- These relations are called transitive.

$\boldsymbol{R}$ on $\boldsymbol{A}$ is transitive $\Leftrightarrow$

$$(\forall \boldsymbol{a})(\forall \boldsymbol{b})(\forall \boldsymbol{c})\big((\boldsymbol{a} \in \boldsymbol{A} \land \boldsymbol{b} \in \boldsymbol{A} \land \boldsymbol{c} \in \boldsymbol{A} \land \boldsymbol{aRb} \land \boldsymbol{bRc}) \to \boldsymbol{aRc}\big)$$

Transitivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $\boldsymbol{N_A}$
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Transitivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$ ☺

- $\boldsymbol{N_A}$ ☺

- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$ ☹
  - $\langle 1,2\rangle \in R, \langle 2,3\rangle \in R$ but $\langle 1,3\rangle \notin R$

Reflexivity, Symmetry and Transitivity

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are reflexive, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cup \boldsymbol{R_2}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also reflexive

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are symmetric, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cup \boldsymbol{R_2}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also symmetric

- If $\boldsymbol{R_1}, \boldsymbol{R_2}$ are transitive, then $\boldsymbol{R_1^{-1}}, \boldsymbol{R_1} \cap \boldsymbol{R_2}$ are also transitive

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are symmetric, then $R_1 \cup R_2$ are also symmetric.

Proof:

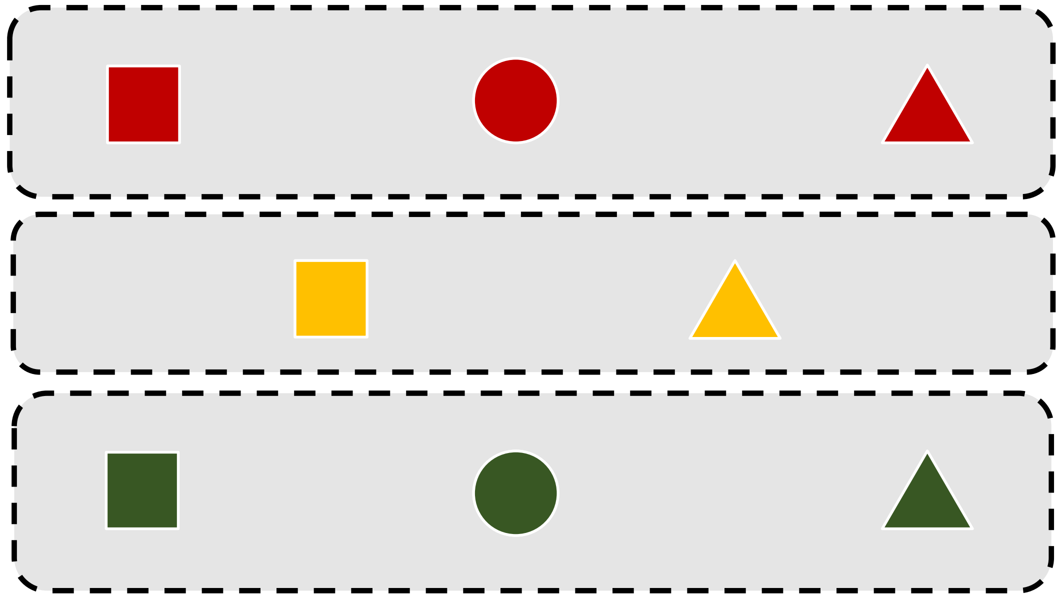
$\langle x, y \rangle \ \in R_1 \cup R_2$
$\Leftrightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2$
$\Leftrightarrow \langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2$
$\Leftrightarrow \langle y, x \rangle \in R_1 \cup R_2$

Q.E.D

Equivalence Relations(等价关系)

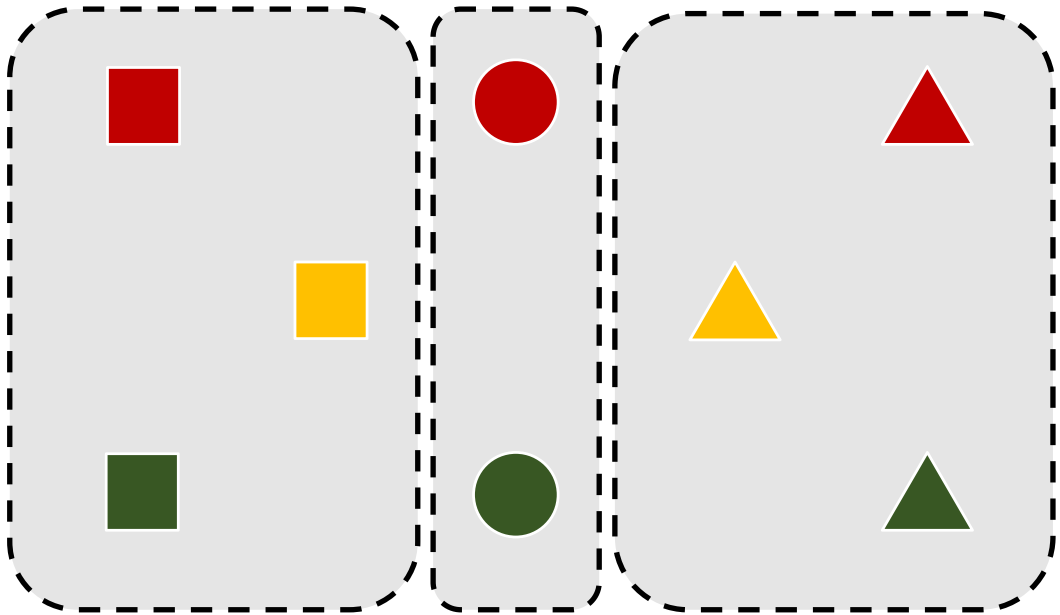
- An **equivalence relation** is a relation that is **reflexive, symmetric and transitive**.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color with y
  - x has the same shape with y

Partitions



aRb if a has the color with b

Partitions



aRb if a has the **shape** with b

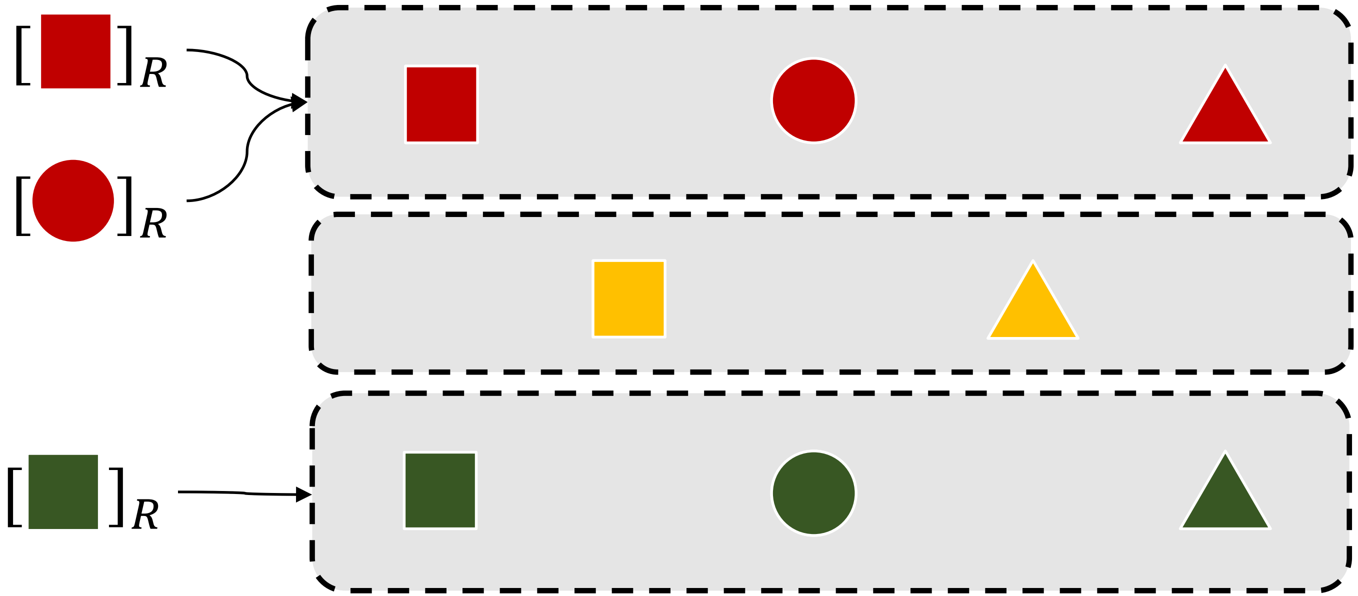
Equivalence Classes(等价类)

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, for any $\boldsymbol{x} \in \boldsymbol{A}$, the **equivalence class of $\boldsymbol{x}$** is the set:

$$[\boldsymbol{x}]_{\boldsymbol{R}} = \{\boldsymbol{y} | \boldsymbol{y} \in \boldsymbol{A} \wedge \boldsymbol{xRy}\}$$

- Intuitively, the set $[\boldsymbol{x}]_{\boldsymbol{R}}$ contains all elements of $\boldsymbol{A}$ that are related to $\boldsymbol{x}$ by relation $\boldsymbol{R}$.

Partitions



aRb if a has the color with b

Quotient Set(商集)

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, the quotient set of $\boldsymbol{A}$ is set of all the disjoint equivalence classes of $\boldsymbol{A}$, donated by $\boldsymbol{A}/\boldsymbol{R}$:

$$\boldsymbol{A}/\boldsymbol{R} = \{\boldsymbol{y} | (\exists \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} = [\boldsymbol{x}]_{\boldsymbol{R}})\}$$

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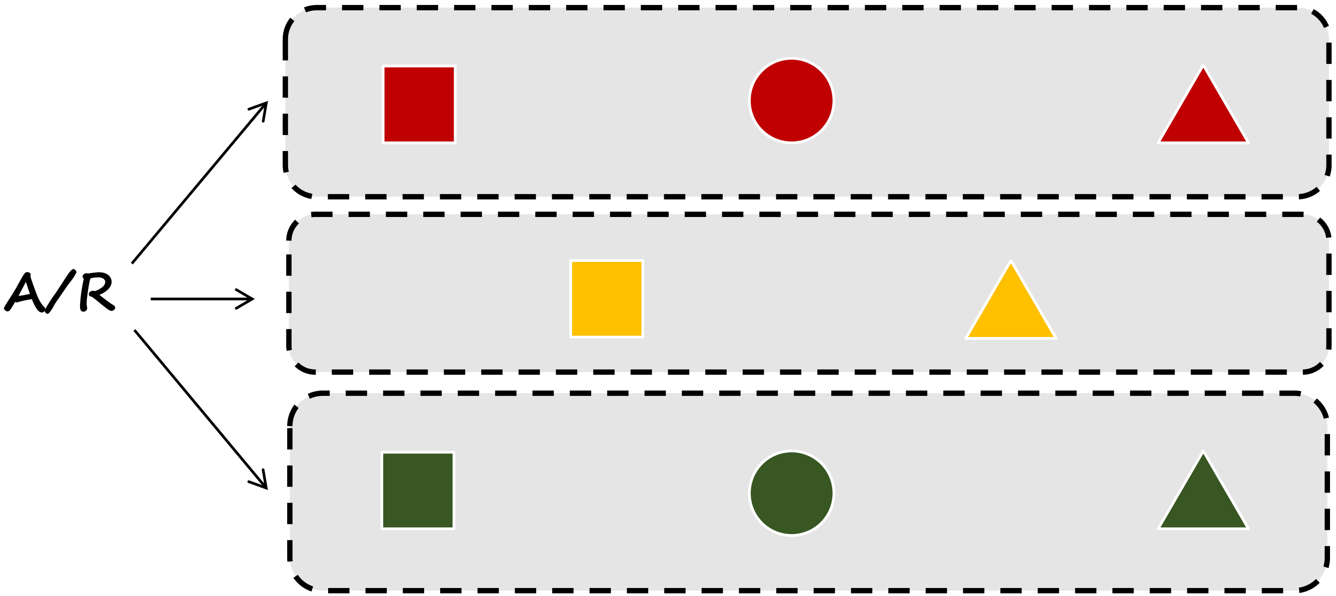
- For $\boldsymbol{R} = \{\langle x, y\rangle | x \equiv_3 y\}$ on set $\boldsymbol{A} = \{1,2,3 \ldots 8\}$
  - $\boldsymbol{A}/\boldsymbol{R} = \{[1]_{\boldsymbol{R}}, [2]_{\boldsymbol{R}}, [3]_{\boldsymbol{R}}\} = \{\{1,4,7\}, \{2,5,8\}, \{3,6\}\}$

Partitions



aRb if a has the color with b

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{R}$ is a partition of $\boldsymbol{A}$

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A/R}$ is a partition of $\boldsymbol{A}$

Proof:

For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}, [\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A/R}$:

Definition of Partition

$\emptyset \notin \boldsymbol{A/R}$

$(\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A/R} \rightarrow \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\cup \boldsymbol{A/R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A/R} \land \boldsymbol{y} \in \boldsymbol{A/R} \land \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{R}$ is a partition of $\boldsymbol{A}$

Proof:

For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}$, $[\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A}/\boldsymbol{R}$:

$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ because $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}}$

Definition of Partition

$\emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$(\forall \boldsymbol{\pi}_0)(\boldsymbol{\pi}_0 \in \boldsymbol{A}/\boldsymbol{R} \rightarrow \boldsymbol{\pi}_0 \subseteq \boldsymbol{A})$

$\cup \boldsymbol{A}/\boldsymbol{R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{y} \in \boldsymbol{A}/\boldsymbol{R} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A/R}$ is a partition of $\boldsymbol{A}$

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ cause $x \in [x]_R$ $\longrightarrow$ $\emptyset \notin A/R$

$[x]_R \subseteq A$ $\longrightarrow$ $(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$\cup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{R}$ is a partition of $\boldsymbol{A}$

Proof:

For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}$, $[\boldsymbol{x}]_{\boldsymbol{R}}, [\boldsymbol{y}]_{\boldsymbol{R}} \in \boldsymbol{A}/\boldsymbol{R}$:

Definition of Partition

$[\boldsymbol{x}]_{\boldsymbol{R}} \neq \emptyset$ cause $\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}} \longrightarrow \emptyset \notin \boldsymbol{A}/\boldsymbol{R}$

$[\boldsymbol{x}]_{\boldsymbol{R}} \subseteq \boldsymbol{A} \longrightarrow (\forall \boldsymbol{\pi_0})(\boldsymbol{\pi_0} \in \boldsymbol{A}/\boldsymbol{R} \rightarrow \boldsymbol{\pi_0} \subseteq \boldsymbol{A})$

$\boldsymbol{x} \in [\boldsymbol{x}]_{\boldsymbol{R}} \longrightarrow \bigcup \boldsymbol{A}/\boldsymbol{R} = \boldsymbol{A}$

$\langle \forall \boldsymbol{x} \rangle \langle \forall \boldsymbol{y} \rangle \langle (\boldsymbol{x} \in \boldsymbol{A}/\boldsymbol{R} \land \boldsymbol{y} \in \boldsymbol{A}/\boldsymbol{R} \land \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation R over a set A, then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$[x]_R \neq \emptyset$ cause $x \in [x]_R$ $\longrightarrow$ $\emptyset \notin A/R$

$[x]_R \subseteq A$ $\longrightarrow$ $(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$x \in [x]_R$ $\longrightarrow$ $\cup A/R = A$

$[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$ $\longrightarrow$ $\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Equivalence Relation and Partition

- For a partition $\boldsymbol{\pi}$ of set $\boldsymbol{A}$, we can define a relation R such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same partition.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | \boldsymbol{\pi_0} \in \boldsymbol{\pi} \wedge \boldsymbol{a} \in \boldsymbol{\pi_0} \wedge \boldsymbol{b} \in \boldsymbol{\pi_0}\}$$

A partition can generate an equivalence relation.

- Given an equivalence relation $\boldsymbol{R}$ over a set $\boldsymbol{A}$, then $\boldsymbol{A/R}$ is a partition of $\boldsymbol{A}$

An equivalence relation can generate a partition.

Equivalence relations precisely capture what it means to be a partition.

Binary relations give us a **common language** to describe **common structures**.

Equivalence Relation In Real Life

- Most modern programming languages include some sort of hash table data structure.
  - **Java: HashMap**
  - **C++: std::unordered_map**
  - **Python: dict**
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relation In Real Life

- "The equals method implements an equivalence relation on non-null object references:
  - It is reflexive: for any non-null reference value x, x.equals⟨x⟩ should return true.
  - It is symmetric: for any non-null reference values x and y, x.equals⟨y⟩ should return true if and only if y.equals⟨x⟩ returns true.
  - It is transitive: for any non-null reference values x, y, and z, if x.equals⟨y⟩ returns true and y.equals⟨z⟩ returns true, then x.equals⟨z⟩ should return true."

Java 8 Documentation

Equivalence Relation In Real Life

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Java 8 Documentation

Equivalence Relation In Real Life

“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements ⟨17.6.3.4⟩ and acts as a hash function for argument values of type *Key*, and by a binary predicate Pred that induces an equivalence relation on values of type *Key*. Additionally, unordered_map and unordered_multimap associate an arbitrary mapped type T with the *Key*.”

C++14 ISO Spec, §23.2.5/3

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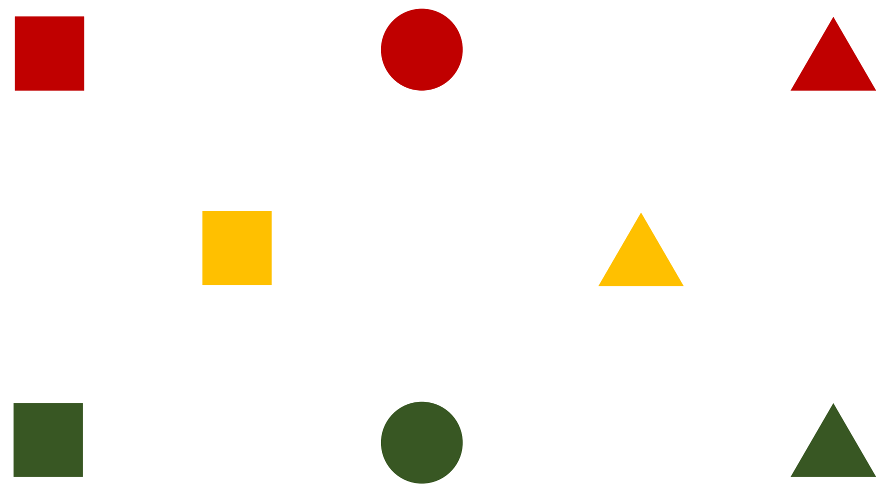
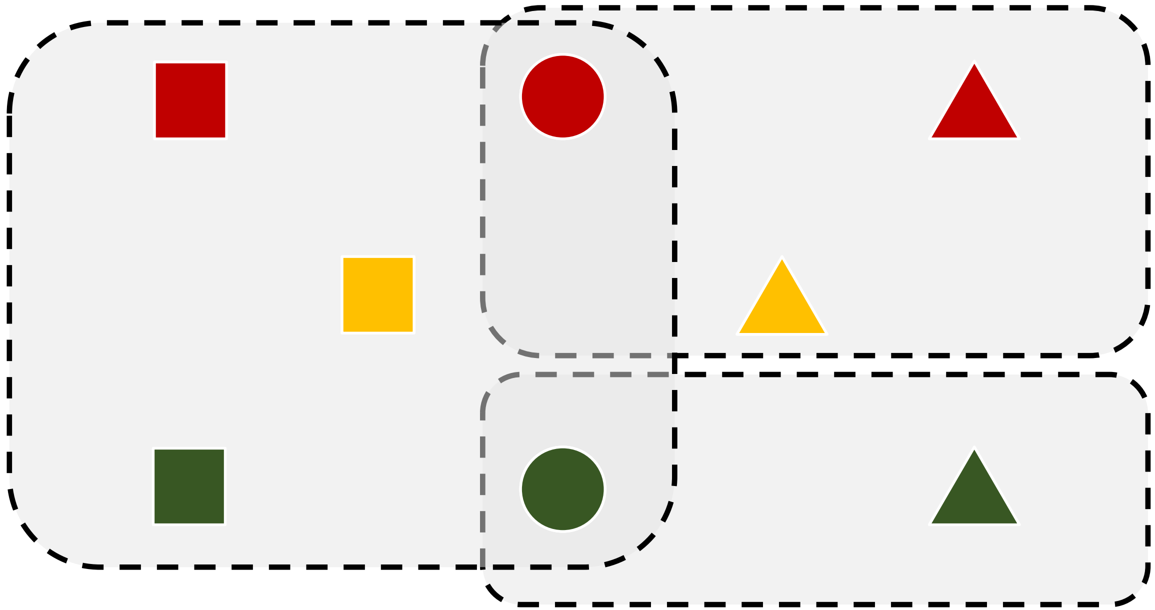
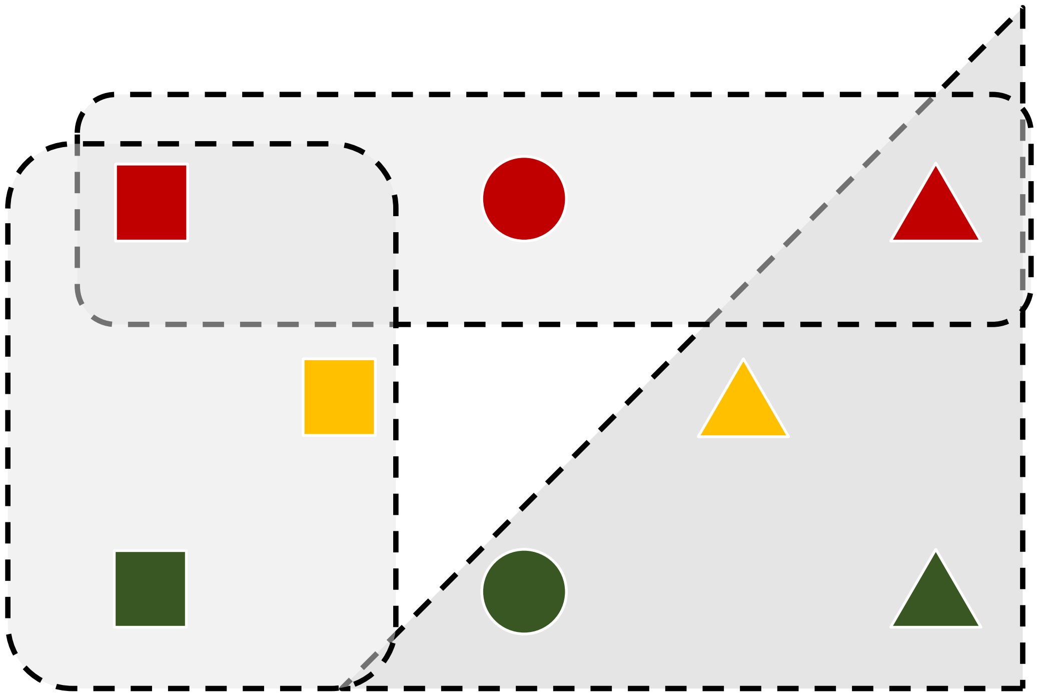
Covers（覆盖）



Covers



Covers



Covers

- A **cover of a set** is a way of splitting the set into **nonempty subsets** so that every element belongs to at least one subset.
- Intuitively, a cover contains pieces that cover every element of a set.
- **A partition is a cover, but a cover is not a partition.**

Covers

- More formally, for a non-empty set $\boldsymbol{A}$, a set $\boldsymbol{C}$ is called the cover of $\boldsymbol{A}$ if:
  - $\emptyset \notin \boldsymbol{C}$
  - $(\forall \boldsymbol{x})(\boldsymbol{x} \in \boldsymbol{C} \rightarrow \boldsymbol{x} \subseteq \boldsymbol{A})$
  - $\bigcup \boldsymbol{C} = \boldsymbol{A}$
  - ~~$(\forall \boldsymbol{x})(\forall \boldsymbol{y})((\boldsymbol{x} \in \boldsymbol{C} \wedge \boldsymbol{y} \in \boldsymbol{C} \wedge \boldsymbol{x} \neq \boldsymbol{y}) \rightarrow \boldsymbol{x} \cap \boldsymbol{y} = \emptyset)$~~

Covers

- For set $A = \{a, b, c, d\}$
  - $\boldsymbol{C_1 = \{\{a\}, \{b, c\}, \{d\}\}}$
  - $\boldsymbol{C_2 = \{\{a, b, c, d\}\}}$
  - $\boldsymbol{C_3 = \left\{\{a, b\}, \{c\}, \{a, d\}\right\}}$
  - $\boldsymbol{C_4 = \{\{a, b, d\}\}}$

valid covers

not a cover

Relation in Cover

- For a cover $\boldsymbol{C}$ of set $\boldsymbol{A}$, we can define a relation $\boldsymbol{R}$ such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same cover block.

$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{C_0})(\boldsymbol{C_0} \in \boldsymbol{C} \wedge \boldsymbol{a} \in \boldsymbol{C_0} \wedge \boldsymbol{b} \in \boldsymbol{C_0})\}$$

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- Then we have:
  - $\boldsymbol{aRa}$ for all $\boldsymbol{a} \in \boldsymbol{A}$
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$
  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

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  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$
  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

An element is in the same block with itself.

If **a** and **b** are in the same block, then **b** and **b** are in the same block too.

If **a** and **b** are in the same block C_0, **b** and **c** are also in the same block $C_1 \neq C_0$, **a** and **b** are not in the same block.

Relation in Cover

- For a cover $\boldsymbol{C}$ of set $\boldsymbol{A}$, we can define a relation $\boldsymbol{R}$ such that $\boldsymbol{aRb}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in a same cover block.

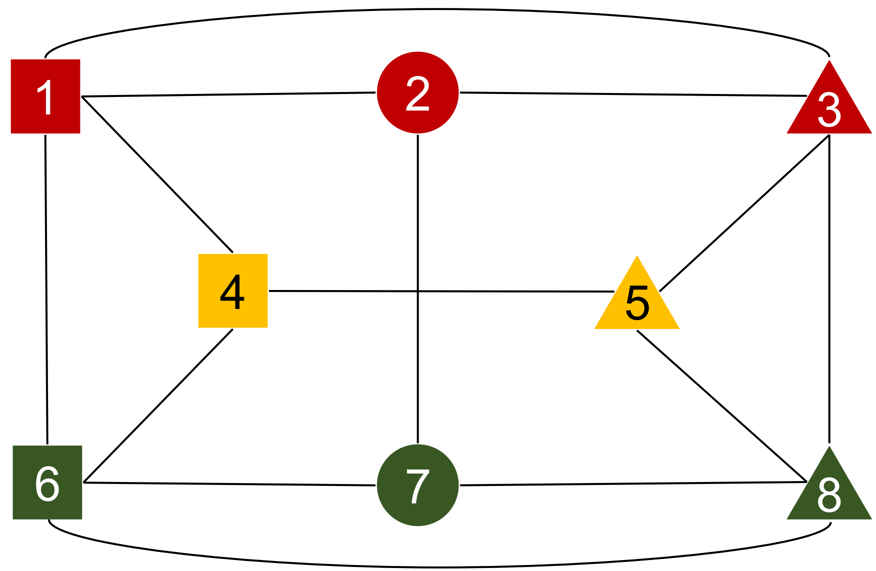
$$\boldsymbol{R} = \{\langle \boldsymbol{a}, \boldsymbol{b} \rangle | (\exists \boldsymbol{C_0})(\boldsymbol{C_0} \in \boldsymbol{C} \wedge \boldsymbol{a} \in \boldsymbol{C_0} \wedge \boldsymbol{b} \in \boldsymbol{C_0})\}$$

- Then we have:
  - $\boldsymbol{aRa}$ for all $\boldsymbol{a} \in \boldsymbol{A}$ — Reflexivity
  - If $\boldsymbol{aRb}$, then $\boldsymbol{bRa}$ — Symmetry
  - ~~If $\boldsymbol{aRb}$ and $\boldsymbol{bRc}$, then $\boldsymbol{aRc}$~~

Compatible Relations (相容关系)

- An **compatible relation** is a relation that is **reflexive and symmetric**.
- Some examples:
  - x has at least one same letter with y
  - $x \cap y \neq \emptyset$
  - x has same color or same shape with y

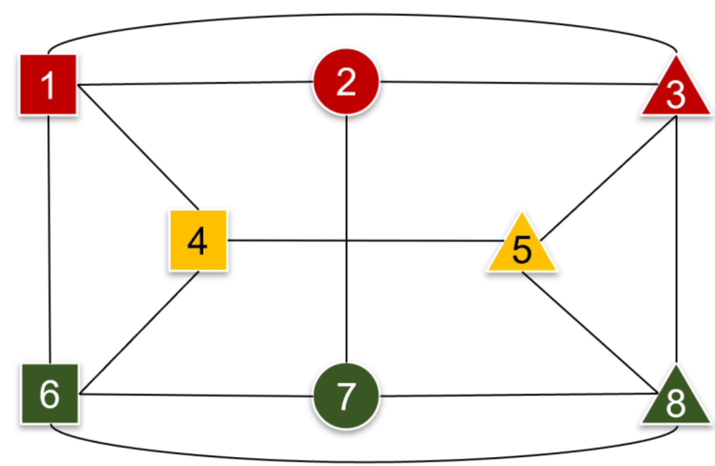
Compatible Relation



xRy if x has same color or same shape with y

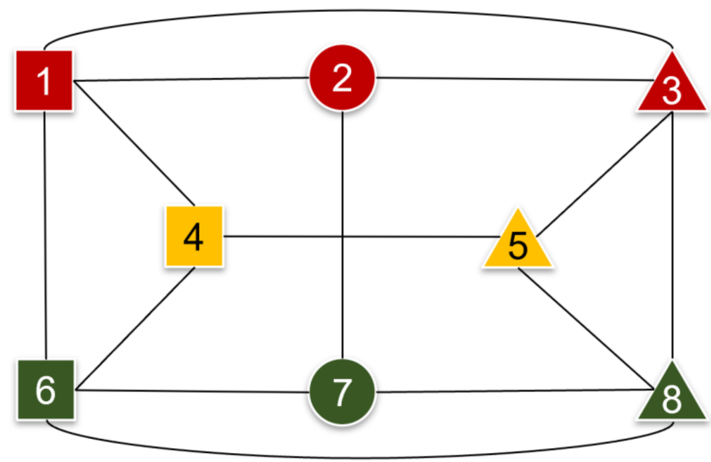
Compatible Class（相容类）

- Given a compatible relation A, C is called **compatible class** if:
  - $C \subseteq A$
  - $(x \in A) \wedge (\forall y)((y \in C) \to xRy)$
- Examples:
  - $\{1,4,6\}$
  - $\{1,4\}$
  - $\{3,8\}$
  - …



Maximal Compatible Class(极大相容类)

- A compatible class C is called **maximal compatible class** if C is not the subset of any compatible class, donate by C_R.
- Examples:
  - $\{1,4,6\}$
  - $\{1,2,3\}$
  - $\{2,7\}$
  - $\{3,5,8\}$
  - …

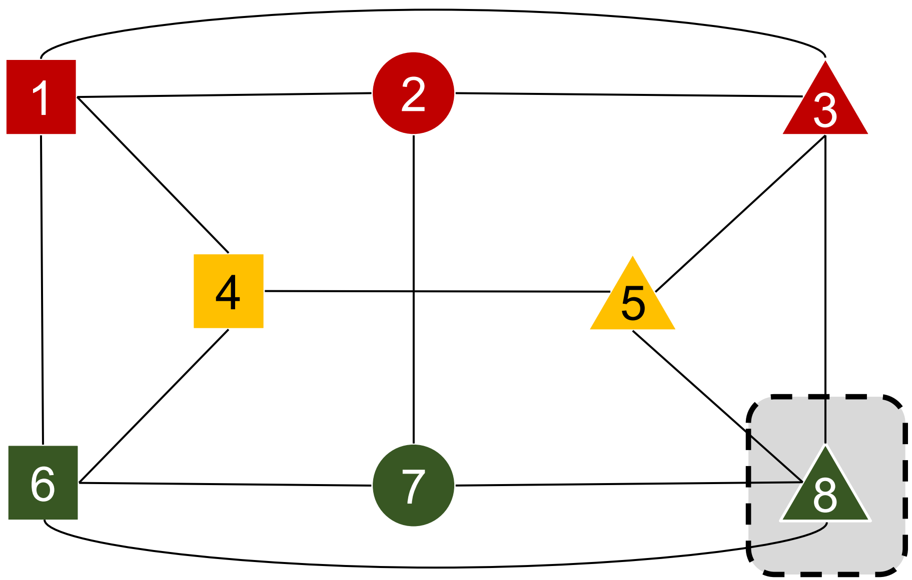


Maximal Compatible Class

For any compatible class C on set $\boldsymbol{A}$, there exists a maximal compatible class C_R such that $C \subseteq C_R$

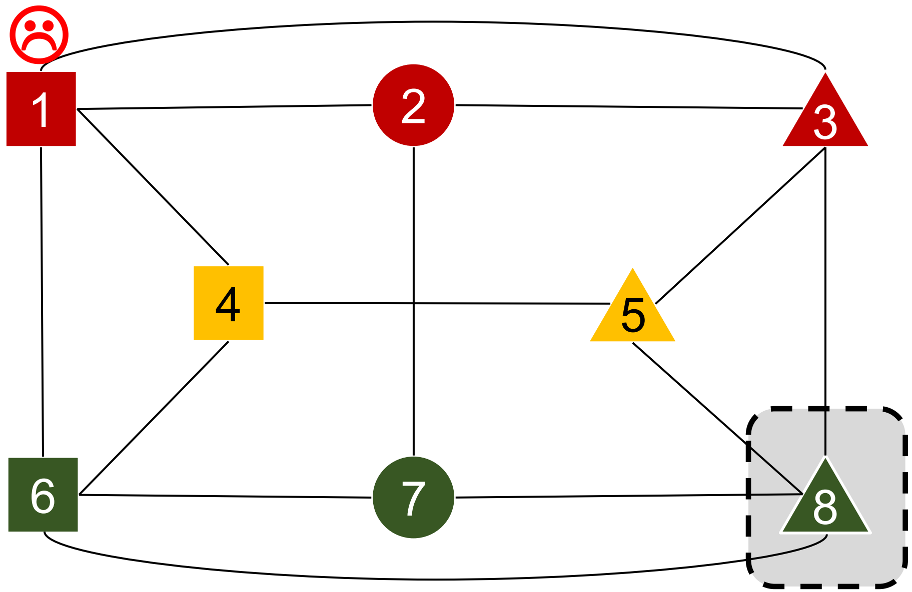
- We can generate a sequence such that $\boldsymbol{C_0} \subseteq \boldsymbol{C_1} \subseteq \boldsymbol{C_2} \ldots \subseteq \boldsymbol{C_R}$
  - $\boldsymbol{C_0} = \boldsymbol{C}$
  - $\boldsymbol{C_{i+1}} = \boldsymbol{C_i} \cup \{\boldsymbol{a_j}\}$, where j is the minimal index such that $\boldsymbol{a_j} \notin \boldsymbol{C_i} \wedge (\boldsymbol{x} \in \boldsymbol{C_i} \rightarrow \boldsymbol{a_j R x})$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

Maximal Compatible Class



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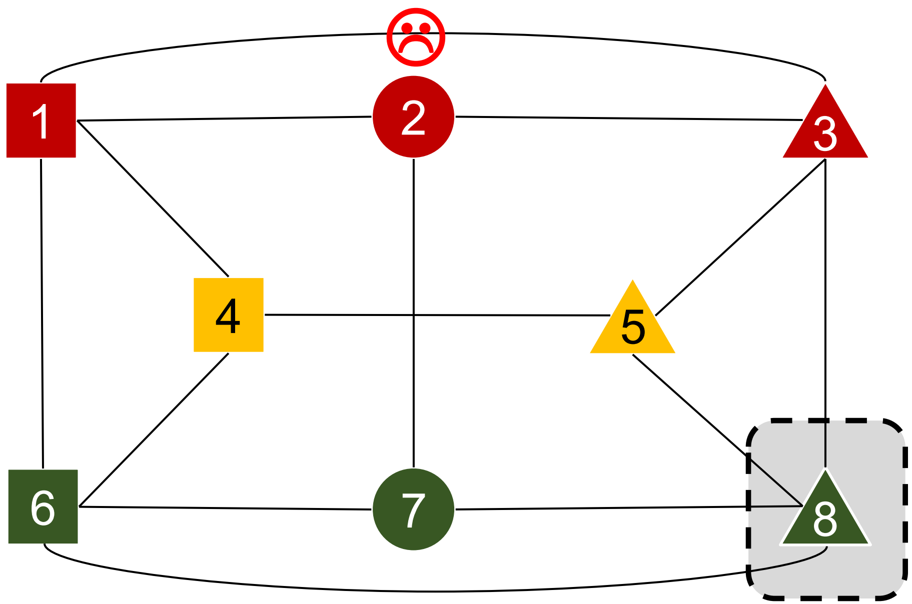

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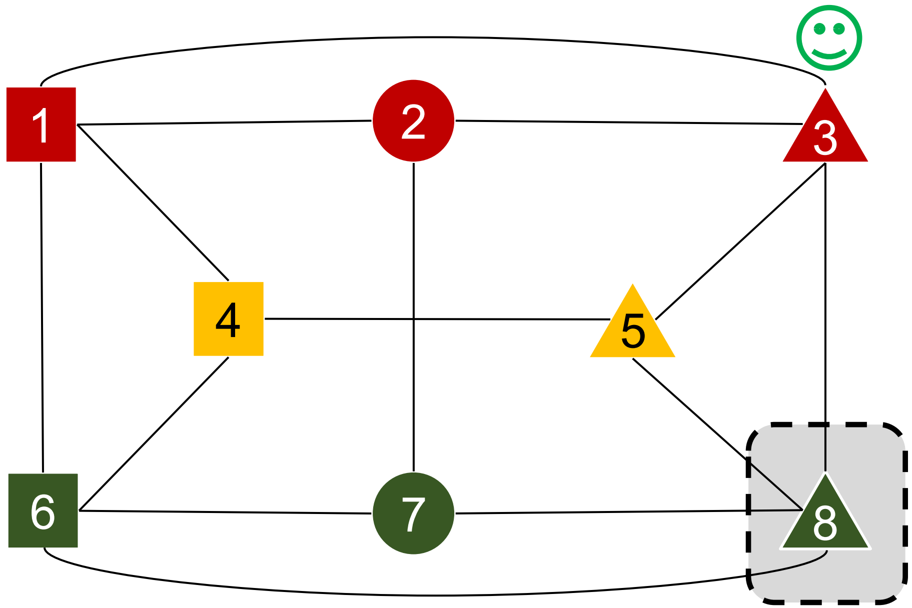

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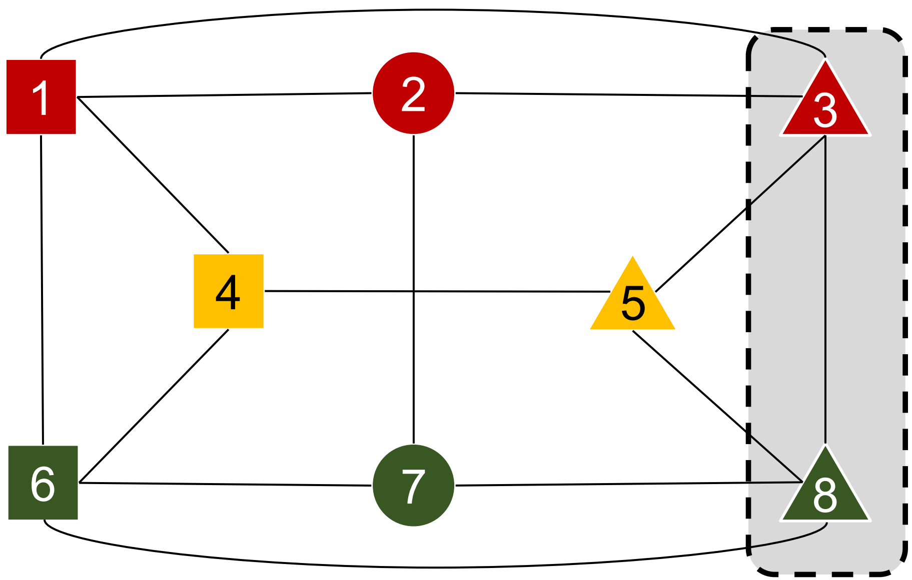
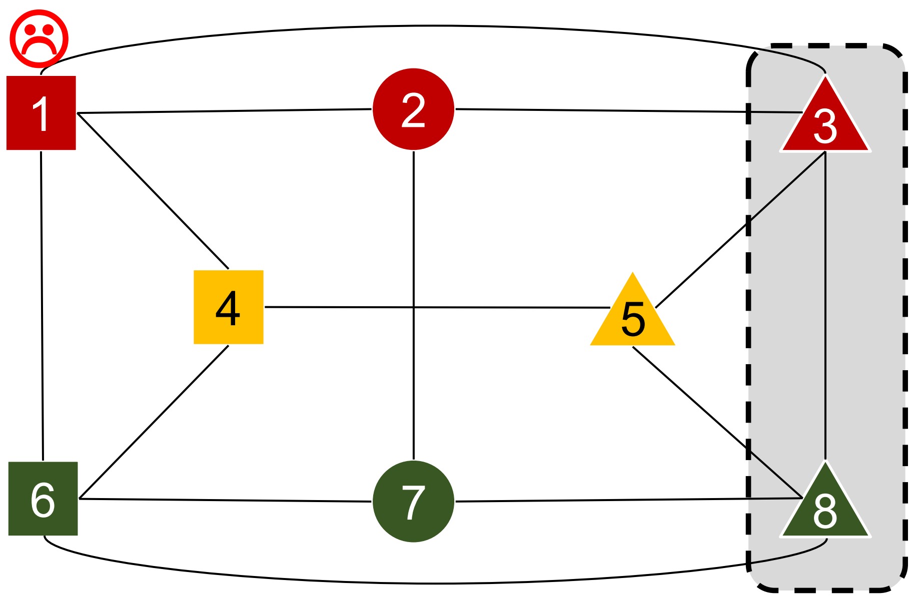


$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

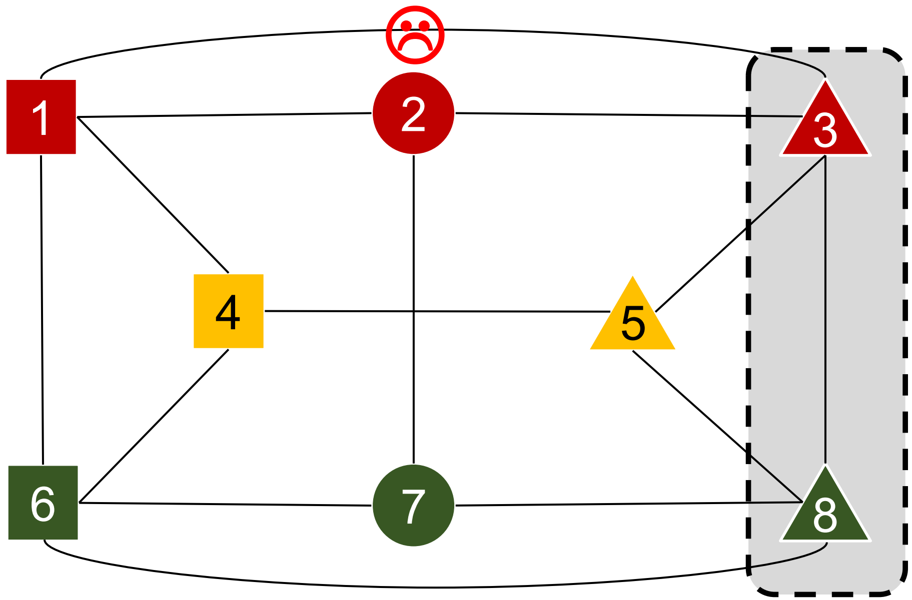
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

Maximal Compatible Class



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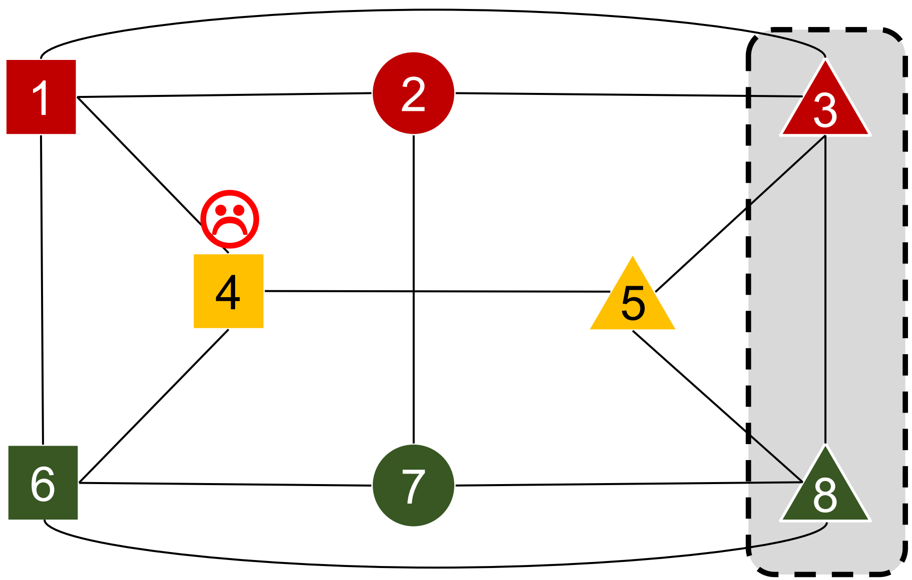
Maximal Compatible Class



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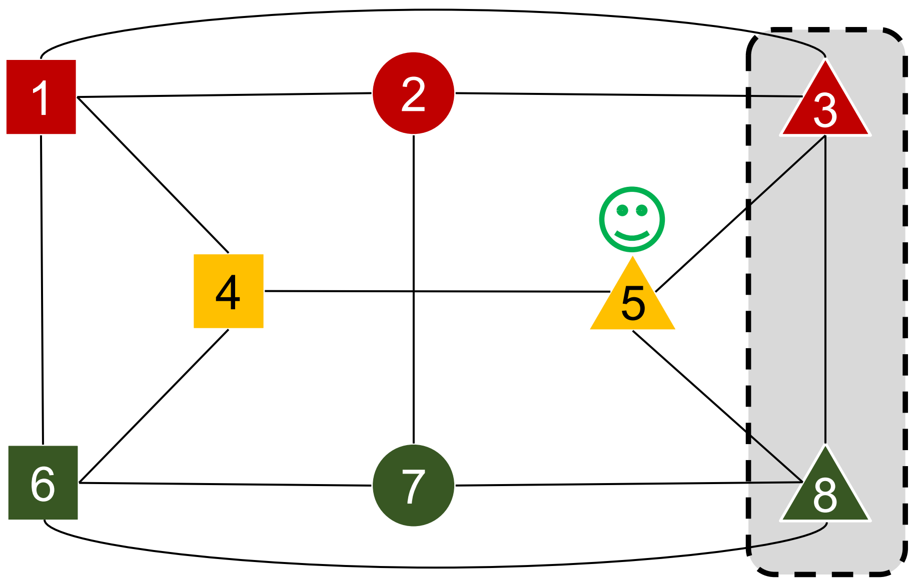
Maximal Compatible Class



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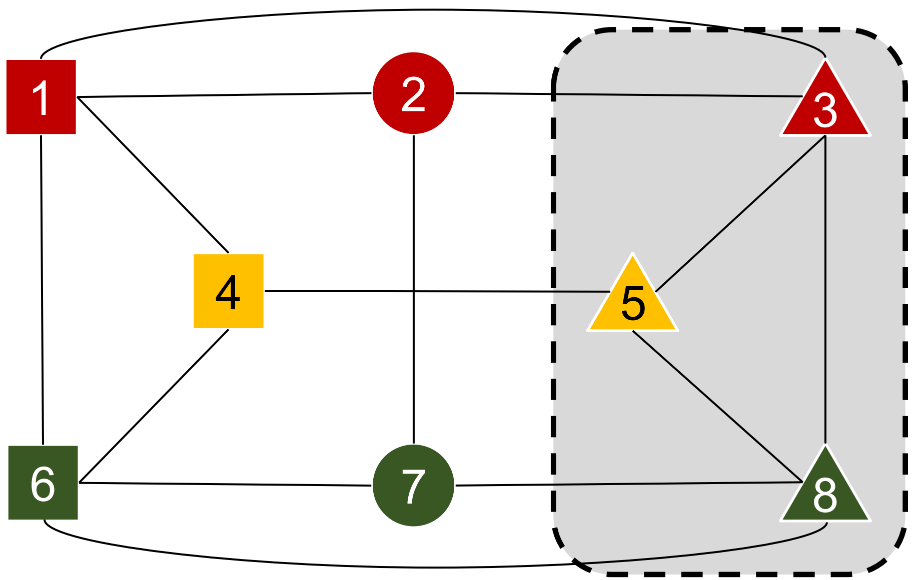
Maximal Compatible Class

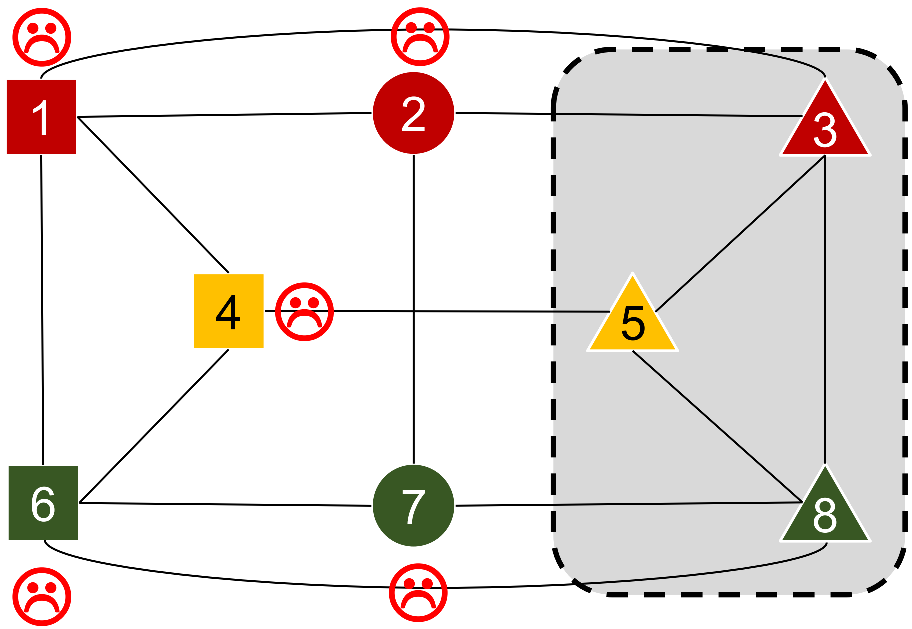


$C_0 = C = \{8\}$

$C_1 = \{3,8\}$

$C_2 = \{3,5,8\}$

Maximal Compatible Class



$C_0 = C = \{8\}$

$C_1 = \{3,8\}$

$C_2 = \{3,5,8\}$

$\boldsymbol{C_R = C_2 = \{3, 5, 8\}}$

Compatible Relation and Covers

- Given a compatible relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$, the set of all maximal compatible classes is a cover of $\boldsymbol{A}$, called the complete cover (完全覆盖) of $\boldsymbol{A}$, donate by $\boldsymbol{C_R(A)}$.
  - $C_R(A)$ is unique.
- A compatible relation of $\boldsymbol{A}$ can generate a complete cover of $\boldsymbol{A}$
- A cover of $\boldsymbol{A}$ can generate a compatible relation of $\boldsymbol{A}$
- Two different covers can generate the same compatible relation.

How we investigate computability?

《数理逻辑与集合论》10.5, 10.8, 11

2.1 Discussion on Problems

Part1. Intro & Set theory(I) : Basics & Formal Language.

Part2. Set Theory(II): Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

2.3 Discussion on computability

Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

An exercise

Let $\boldsymbol{R}$ be the relation on the set of ordered pairs of positive integers such that $\langle(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d})\rangle \in \boldsymbol{R}$ if and only if $a + d = b + c$. Show that $\boldsymbol{R}$ is an equivalence relation.

An exercise

Let $\boldsymbol{R}$ be the relation on the set of ordered pairs of positive integers such that $\langle(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d})\rangle \in \boldsymbol{R}$ if and only if $a + d = b + c$. Show that $\boldsymbol{R}$ is an <u>equivalence relation</u>.

Reflexive, Symmetric and Transitive

An exercise

Proof:

- ***Reflexive:*** $\boldsymbol{xRx}$

  For any integer pair $(\boldsymbol{a}, \boldsymbol{b})$, we have $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$, so $(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{R}(\boldsymbol{a}, \boldsymbol{b})$

- ***Symmetric:*** $\boldsymbol{xRy} \rightarrow \boldsymbol{yRx}$

  For integer pair $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$, if $(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{d})$, we have $\boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c}$. So $\boldsymbol{c} + \boldsymbol{b} = \boldsymbol{d} + \boldsymbol{a}$ also holds, it means $(\boldsymbol{c}, \boldsymbol{d})\boldsymbol{R}(\boldsymbol{a}, \boldsymbol{b})$

- ***Transitive:*** $\boldsymbol{xRy} \wedge \boldsymbol{yRz} \rightarrow \boldsymbol{xRz}$

  For integer pair $(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d})$ and $(\boldsymbol{e}, \boldsymbol{f})$, if $(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{R}(\boldsymbol{c}, \boldsymbol{d})$ and $(\boldsymbol{c}, \boldsymbol{d})\boldsymbol{R}(\boldsymbol{e}, \boldsymbol{f})$, we have $\boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c}$ and $\boldsymbol{c} + \boldsymbol{f} = \boldsymbol{d} + \boldsymbol{e}$. Add the two equation we get $\boldsymbol{a} + \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f} = \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d} + \boldsymbol{e} \Rightarrow \boldsymbol{a} + \boldsymbol{f} = \boldsymbol{b} + \boldsymbol{e}$, it means $(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{R}(\boldsymbol{e}, \boldsymbol{f})$

Order Relations

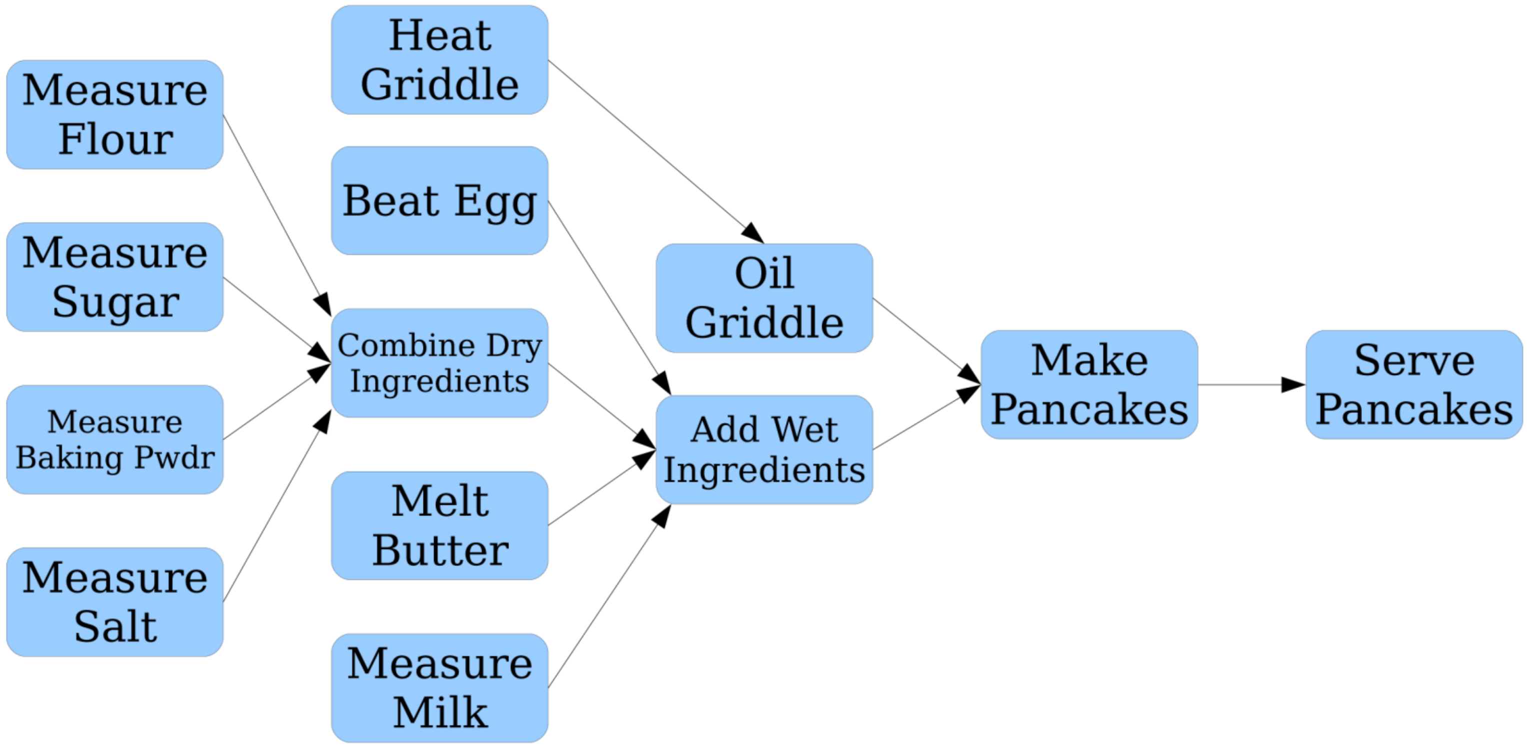
Ordered Objects

words in dictionary

a	bread	dog
about	bring	far
after	but	fly
an	buy	got
ask	cake	Hand
bear	call	it
bed	chair	letter

Ordered Objects

A recipe to make pancake



Ordered Objects

- Take $\leq$ on $\mathbb{R}$ as example. We know that:
  - $x \leq x$ for any $x \in \mathbb{R}$
  - If $x \leq y$ and $y \leq x$,then $x = y$
  - If $x \leq y$ and $y \leq z$,then $x \leq z$

Ordered Objects

- Take $\leq$ on $\mathbb{R}$ as example. We know that:
  - $x \leq x$ for any $x \in \mathbb{R}$ Reflexivity
  - If $x \leq y$ and $y \leq x$,then $x = y$?
  - If $x \leq y$ and $y \leq z$,then $x \leq z$ Transitivity

Anti-symmetry(反对称性)

- Some relation is symmetric only in its diagonal.
- Examples:
  - If $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{y} \leq \boldsymbol{x}$,then $\boldsymbol{x} = \boldsymbol{y}$
  - If $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$, then $\boldsymbol{A} = \boldsymbol{B}$
- Relations of this sort are called antisymmetric.

$$\boldsymbol{R} \text{ on } \boldsymbol{A} \text{ is antisymmetric} \Leftrightarrow$$

$$(\forall \boldsymbol{a})(\forall \boldsymbol{b})((\boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \boldsymbol{A} \wedge \boldsymbol{aRb} \wedge \boldsymbol{bRa}) \rightarrow \boldsymbol{a} = \boldsymbol{b})$$

Anti-symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{N_A}$
- $\boldsymbol{E_A}$ when $|\boldsymbol{A}| > 1$
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Anti-symmetry : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{N_A}$ ☺
- $\boldsymbol{E_A}$ when $|\boldsymbol{A}| > 1$ ☹
- $\boldsymbol{R} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$ ☹
  - $1R2$ and $2R1$

Partial-order Relation(偏序关系)

- An **partial-order relation** is a relation that is **reflexive, anti-symmetric and transitive**.
- Some example:
  - $x \leq y$
  - $x \subseteq y$
  - x can be divided by y (on $\mathbb{N}^+$)
- A set $\boldsymbol{A}$ together with a partial ordering $\boldsymbol{R}$ is called a partially ordered set, or poset(偏序集), and is denoted by $\langle \boldsymbol{A}, \boldsymbol{R} \rangle$.
  - $\langle \mathbb{N}, \leq \rangle$,$\langle \mathcal{P}(A), \subseteq \rangle$ are both poset.

Ordered Objects

- Take $<$ on $\mathbb{R}$ as example. We know that:
  - $x \nless x$ for any $x \in \mathbb{R}$
  - If $x < y$ and $y < z$,then $x < z$

Ordered Objects

- Take $<$ on $\mathbb{R}$ as example. We know that:
  - $x \nless x$ for any $x \in \mathbb{R}$?
  - If $x < y$ and $y < z$,then $x < z$ Transitivity

Anti-reflexivity(反自反性)

- Some relation never holds from any element to itself
- Examples:
  - $\boldsymbol{A} \not\subset \boldsymbol{A}$ for any set $\boldsymbol{A}$
  - $\boldsymbol{x} \nless \boldsymbol{x}$ for any $\boldsymbol{x}$
- Relations of this sort are called anti-reflexive or irreflexive:

$$\boldsymbol{R} \text{ on } \boldsymbol{A} \text{ is anti-reflexive} \Leftrightarrow (\forall \boldsymbol{a})(\boldsymbol{a} \in \boldsymbol{A} \rightarrow \boldsymbol{a} \not\boldsymbol{R} \boldsymbol{a})$$

Anti-reflexivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$
- $\boldsymbol{N_A}$
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$

Anti-reflexivity : Example

- $\boldsymbol{I_A}$ and $\boldsymbol{E_A}$ ☹
- $\boldsymbol{N_A}$ ☺
- $\boldsymbol{R} = \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,3\rangle\}$ on $\{1,2,3\}$ ☹
  - $3R3$

Reflexivity and Anti-reflexivity

- Reflexivity and anti-reflexivity are not negations of one another!
- Reflexivity:

$$(\forall \boldsymbol{a})(a \in A \to a\boldsymbol{R}a)$$

- Negation of Reflexivity(Non-reflexivity,非自反性):

$$(\exists \boldsymbol{a})(a \in a \to a\not{\boldsymbol{R}}a)$$

- Anti-reflexivity:

$$(\forall \boldsymbol{a})(a \in A \to a\not{\boldsymbol{R}}a)$$

Quasi-order Relation(拟序关系)

- An **quasi-order relation, or strict order relation** is a relation that is **anti-reflexive, transitive**.
- Some example:
  - $x < y$
  - $x \subset y$
  - x is taller than y

Quasi-order Relation

Quasi-order relation is anti-symmetric.

Quasi-order Relation

Quasi-order relation is anti-symmetric.

Anti-reflexivity
Transitivity $\Rightarrow$ Anti-symmetric

Quasi-order Relation

Quasi-order relation is anti-symmetric.

$$\begin{matrix} \text{Anti-reflexivity} \\ \text{Transitivity} \end{matrix} \Rightarrow \text{Anti-symmetric}$$

Proof:

For a quasi-order relation $\boldsymbol{R}$ on $\boldsymbol{A}$, assume that $\boldsymbol{R}$ is not anti-symmetric, then there exits $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{xRy} \wedge \boldsymbol{yRx} \wedge \boldsymbol{x} \neq \boldsymbol{y}$.

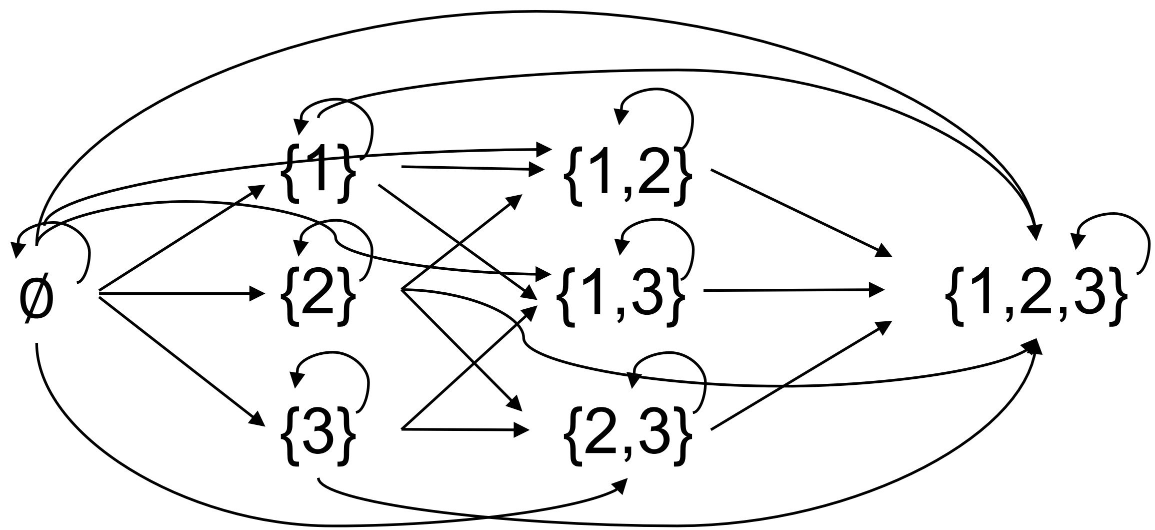
From transitivity and $\boldsymbol{xRy} \wedge \boldsymbol{yRx}$ we have $\boldsymbol{xRx}$, but $\boldsymbol{x}\not{\boldsymbol{R}}\boldsymbol{x}$ as $\boldsymbol{R}$ is anti-reflexivity.

Contradiction.

Partial Order and Quasi Order

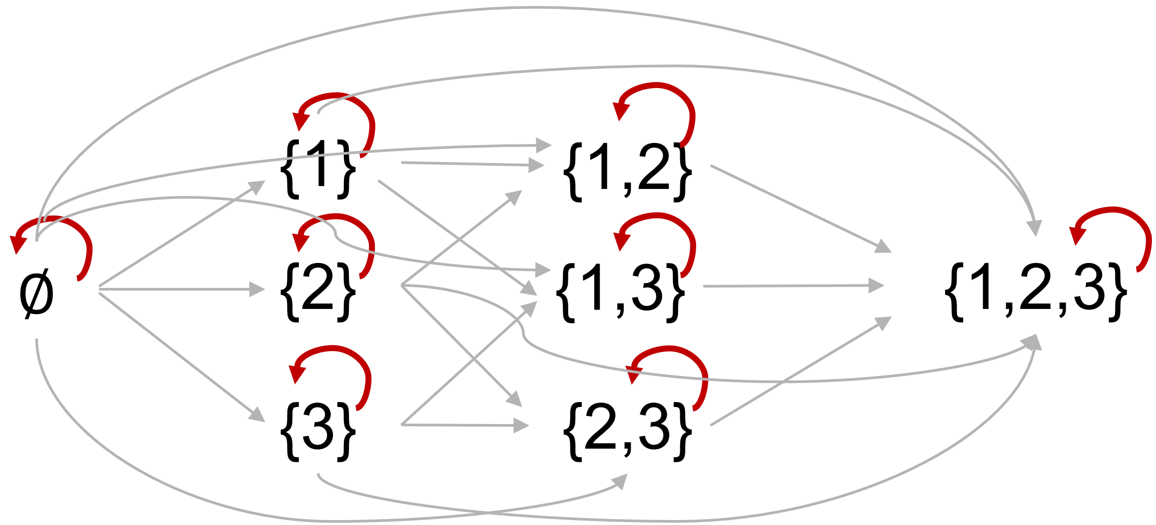
- Often donate as $\leq$ and $<$
  - Not necessarily mean numerical operation.
- Quasi order $\boldsymbol{R}$ on $\boldsymbol{A}$, $\boldsymbol{R} \cup \boldsymbol{R_0}$ is a partial order
  $"<" + "=" \rightarrow " \leq "$
- Partial order $\boldsymbol{R}$ on $\boldsymbol{A}$, $\boldsymbol{R} - \boldsymbol{R_0}$ is a quasi order
  $" \leq " - "=" \rightarrow " < "$

$\langle \mathcal{P}(A), \subseteq \rangle$, A={1,2,3}



Too complicated!!!

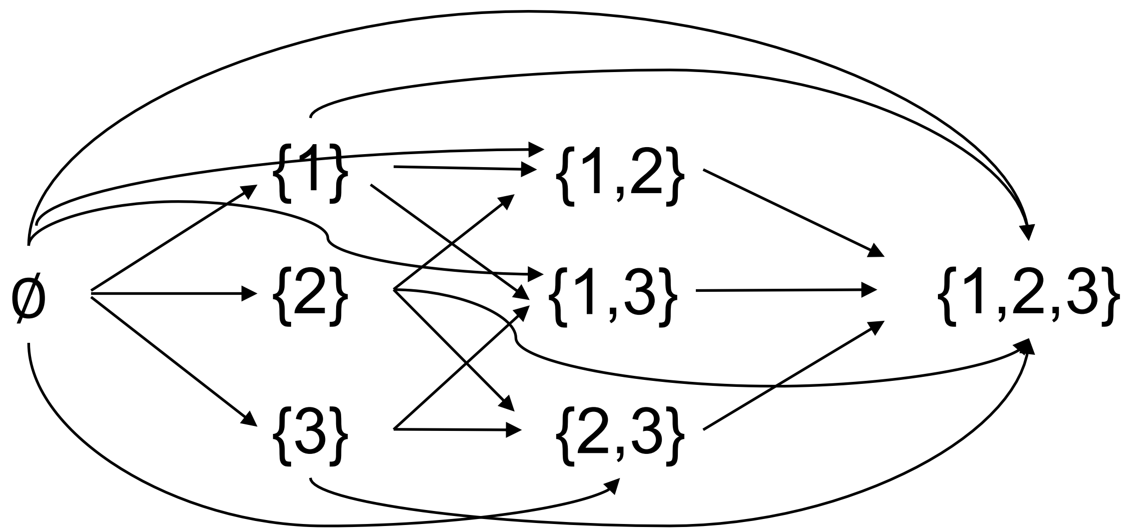
$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$



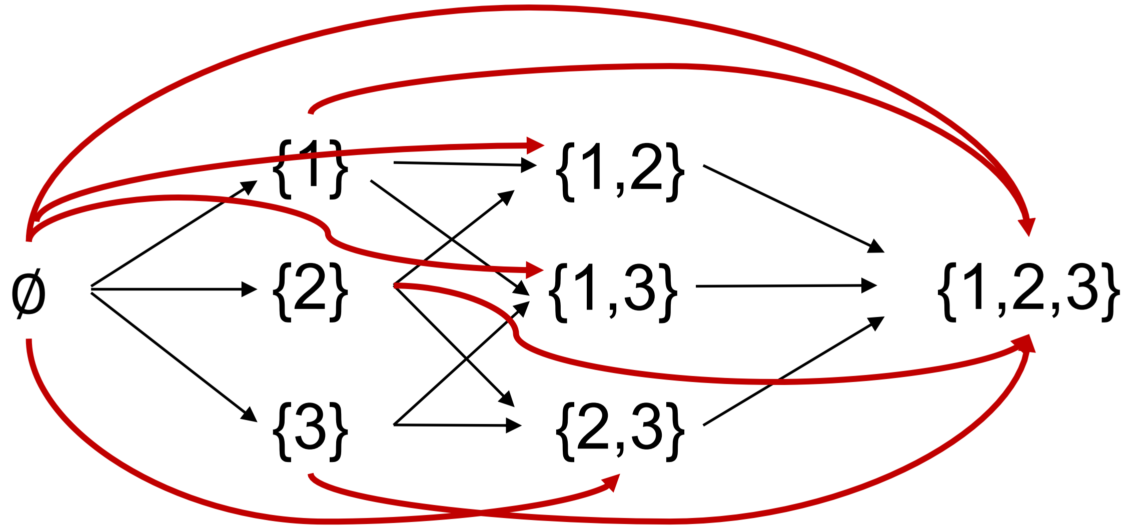
Reflexivity: Every object has a self-circle.

Drop it.

$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$



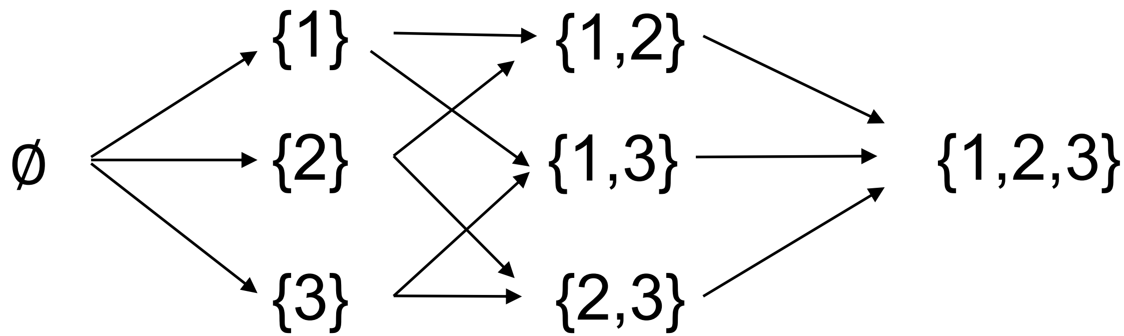
$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$



Transitivity: Some relation can be derived from "smaller" relation

Drop it.

$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$



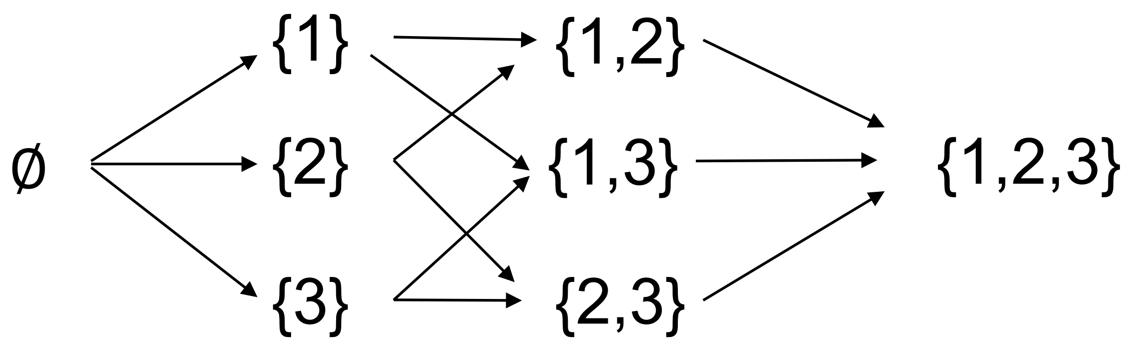
Cover Relation

- For poset $\langle \boldsymbol{A}, \leq \rangle$, if $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{A}, \boldsymbol{x} \leq \boldsymbol{y}, \boldsymbol{x} \neq \boldsymbol{y}$, and there doesn't exist $\boldsymbol{z} \in \boldsymbol{A}$ such that $\boldsymbol{x} \leq \boldsymbol{z}$ and $\boldsymbol{z} \leq \boldsymbol{y}$, we call that **$\boldsymbol{y}$ covers $\boldsymbol{x}$**.
- $\boldsymbol{y}$ covers $\boldsymbol{x}$ means that $\boldsymbol{y}$ is exactly greater than $\boldsymbol{x}$ and there's no elements between them.
- We can define a **cover relation** on $\boldsymbol{A}$:

$$cov(A) = \{\langle \boldsymbol{x}, \boldsymbol{y} \rangle | \boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} \in \boldsymbol{A} \wedge (\boldsymbol{y}\ covers\ \boldsymbol{x})\}$$

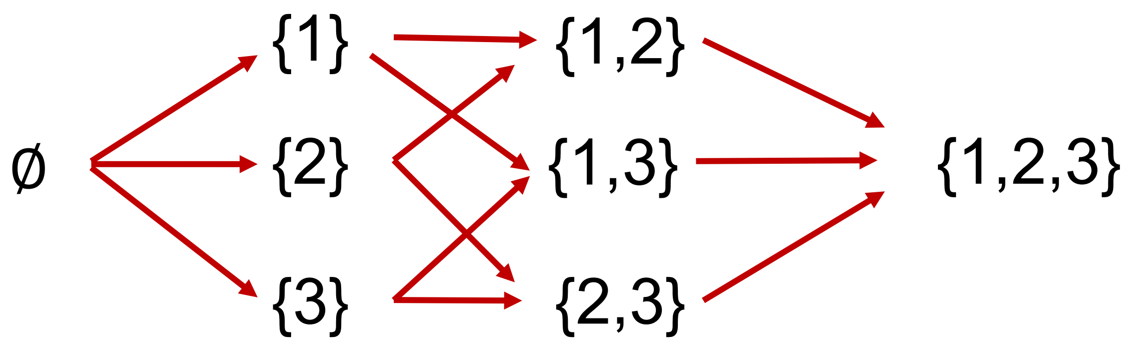
$$\langle \mathcal{P}(A), \subseteq \rangle,\ A=\{1,2,3\}$$

$$cover(\mathcal{P}(A))$$



$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$

$cover(\mathcal{P}(A))$

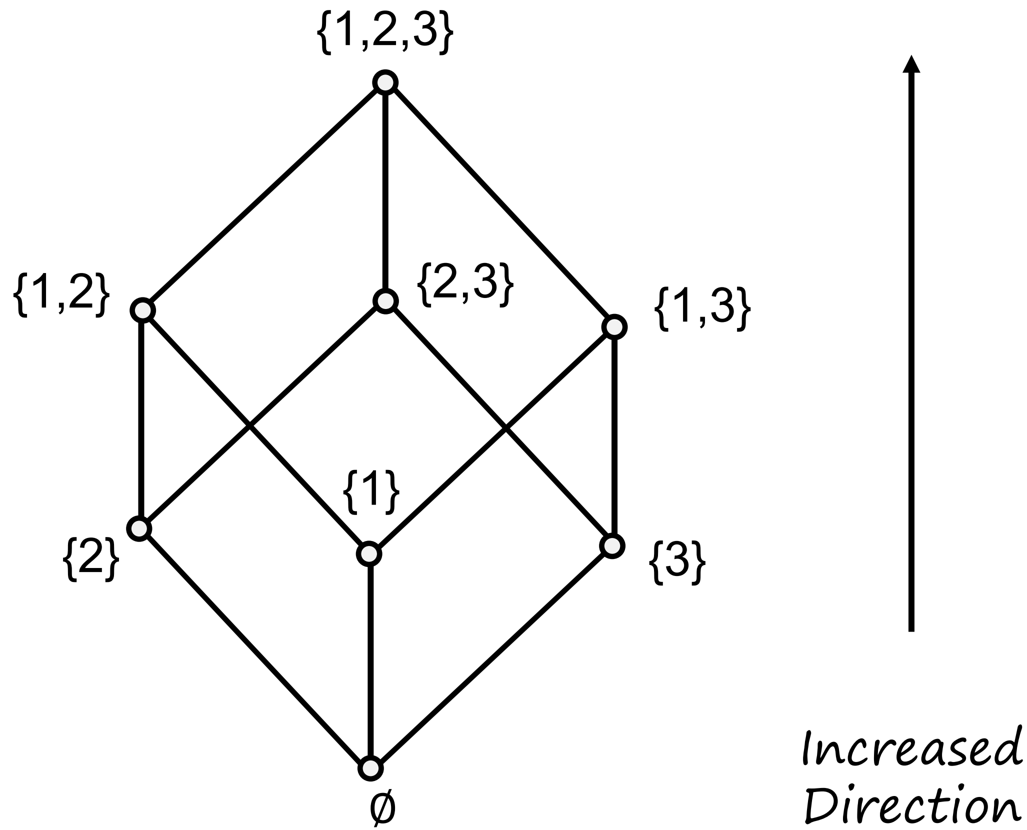


All directed edge are towards the same way

Choose a direction as increased direction.

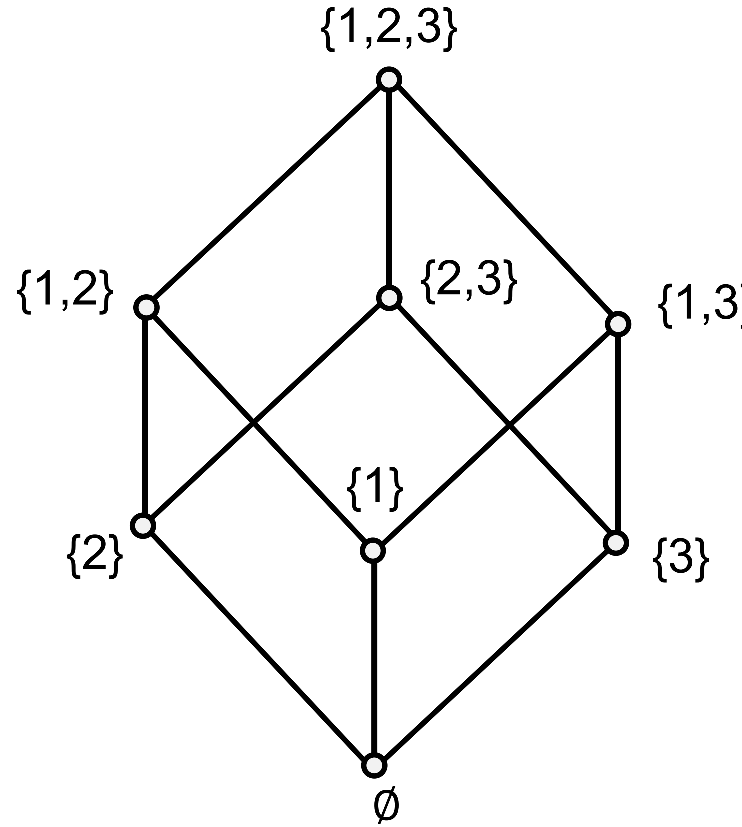
$\langle \mathcal{P}(A), \subseteq \rangle$, $A=\{1,2,3\}$

$cover(\mathcal{P}(A))$



Hasse Diagram(哈斯图)

- For poset $\langle A, \leq \rangle$:
  - Every vertex is an element
  - If $x \leq y \wedge x \neq y$, then vertex y is above vertex x
  - If $\langle x, y \rangle \in cov(A)$, then there's an undirected edge between x and y



Hasse Diagram

- Define the divisibility relation on $A = \{1,2,3,4,6,12\}$:

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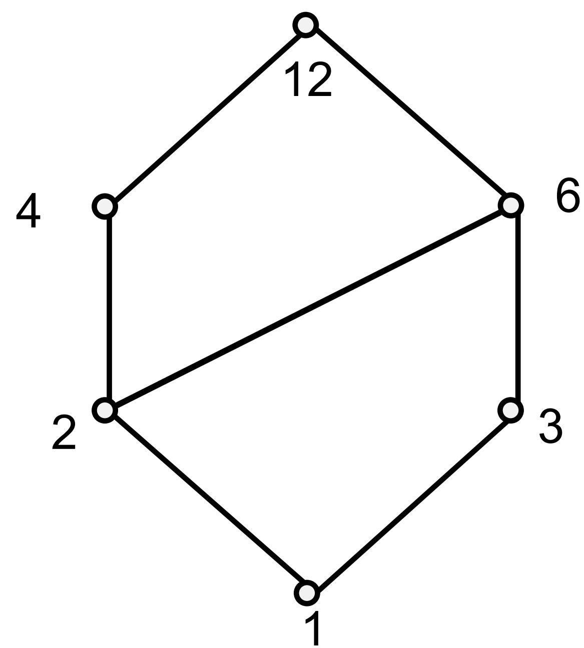

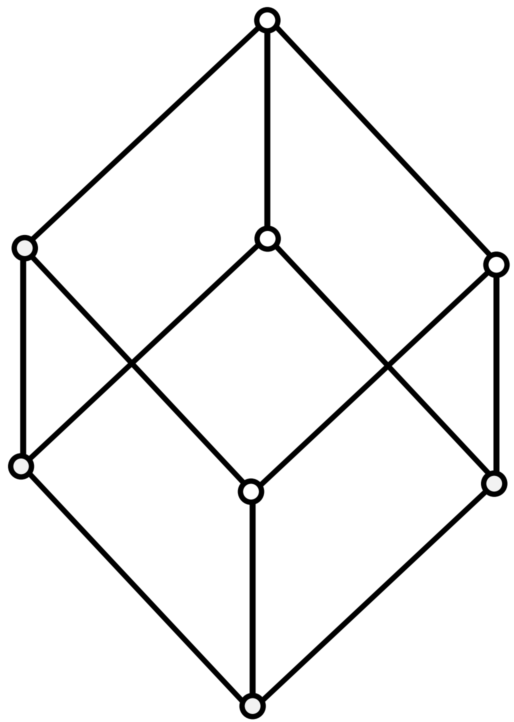

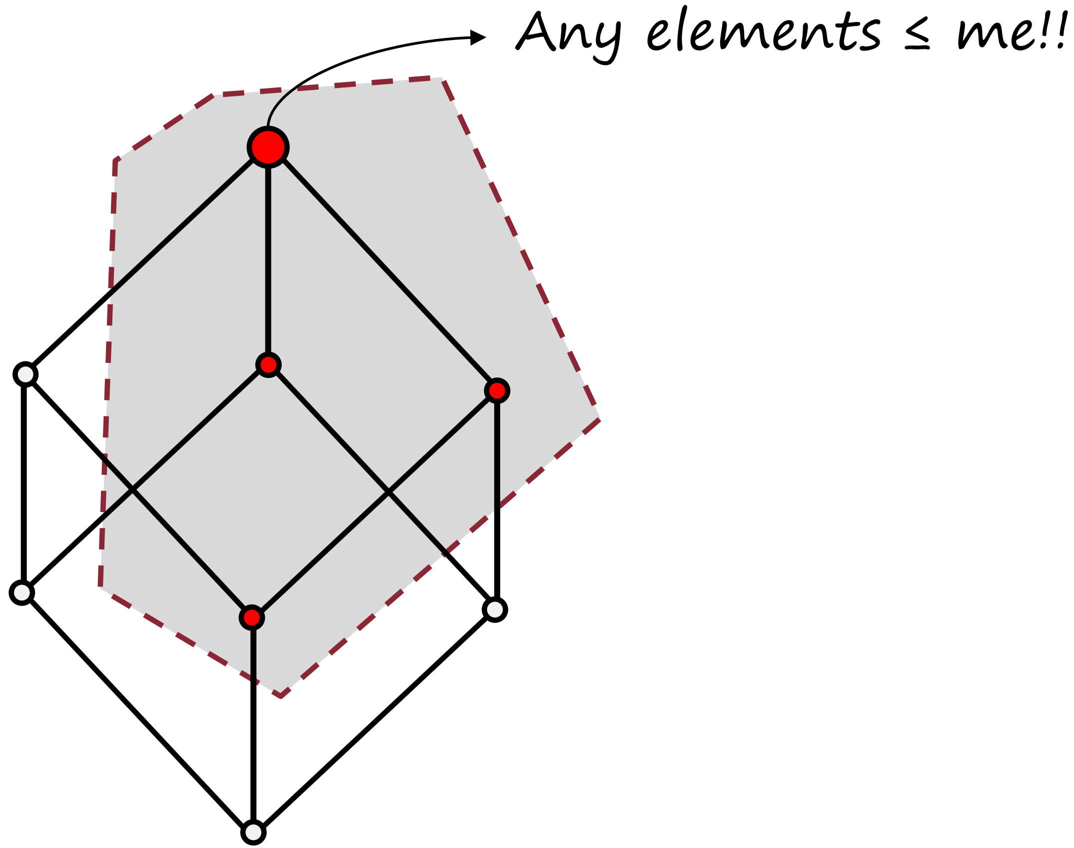
Any elements ≤ me!!!